Feedback Linearization for Double Pendulum Control

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Abstract

This report will show the procedure of how to control a double pendulum using feedback linearization. The report will start from the derivation of the double pendulum system, and them derive the feedback linearization controller. Finally simulation results will be shown.

1 Double Pendulum Dynamics

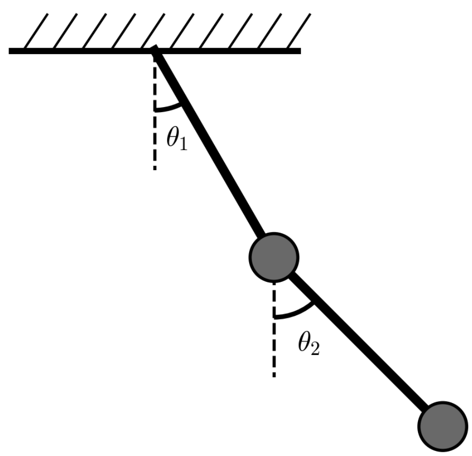

Figure 1: **Illustration of double pendulum**

For the double pendulum shown in Figure 1, we can calculate the position of the two masses as

$$p_1 = \begin{pmatrix} \ell_1 \sin(\theta_1) \\ -\ell_1 \cos(\theta_1) \end{pmatrix} \tag{1a}$$

$$p_2 = \begin{pmatrix} \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) \\ -\ell_1 \cos(\theta_1) - \ell_2 \cos(\theta_2) \end{pmatrix} \tag{1b}$$

which by differentiating gives the velocities of the two masses

$$\dot{p}_1 = \begin{bmatrix} \ell_1 \cos(\theta_1)\dot{\theta}_1 \\ \ell_1 \sin(\theta_1)\dot{\theta}_1 \end{bmatrix} \tag{2a}$$

$$\dot{p}_2 = \begin{bmatrix} \ell_1 \cos(\theta_1)\dot{\theta}_1 + \ell_2 \cos(\theta_2)\dot{\theta}_2 \\ \ell_1 \sin(\theta_1)\dot{\theta}_1 + \ell_2 \sin(\theta_2)\dot{\theta}_2 \end{bmatrix}. \tag{2b}$$

We can get its kinetic energy as

$$
\begin{aligned}
T &= \frac{1}{2}m_1\dot{p}_1^2 + \frac{1}{2}m_2\dot{p}_2^2 \\
&= \frac{1}{2}m_1\Big[(\ell_1\cos(\theta_1)\dot{\theta}_1)^2 + (\ell_1\sin(\theta_1)\dot{\theta}_1)^2\Big] + \frac{1}{2}m_2\dot{p}_2^2 \\
&= \frac{1}{2}m_1\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\Big[(\ell_1\cos(\theta_1)\dot{\theta}_1 + \ell_2\cos(\theta_2)\dot{\theta}_2)^2 + (\ell_1\sin(\theta_1)\dot{\theta}_1 + \ell_2\sin(\theta_2)\dot{\theta}_2)^2\Big] \\
&= \frac{1}{2}m_1\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\Big[\ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2))\Big] \\
&= \frac{1}{2}m_1\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\Big[\ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\Big]
\end{aligned}
$$

which gives us

$$
T = \frac{1}{2}(m_1 + m_2)\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\ell_2^2\dot{\theta}_2^2 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2). \tag{3}
$$

From (1) we can easily obtain the potential energy as

$$
V = -m_1 g\ell_1\cos(\theta_1) - m_2 g\big[\ell_1\cos(\theta_1) + \ell_2\cos(\theta_2)\big]. \tag{4}
$$

We can see that both the kinetic and potential energy aligns with what is presented in [1], under different definitions of the ground height. Now that we have T and V, we define the Lagrangian as

$$
\begin{aligned}
L &= T - V \qquad (5) \\
&= \frac{1}{2}(m_1 + m_2)\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\ell_2^2\dot{\theta}_2^2 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + m_1 g\ell_1\cos(\theta_1) \\
&\quad + m_2 g\big[\ell_1\cos(\theta_1) + \ell_2\cos(\theta_2)\big].
\end{aligned}
$$

Using the Euler-Lagrangian equations

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \Gamma \tag{6}
$$

with q being the generalized coordinates and Γ being the generalized force, we can arrive at the dynamics equations of the double pendulum system. Here we define

$$
q = \begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix} \text{ and } \Gamma = \begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix}.
$$

Using this definition we have

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{q}_1} &= (m_1 + m_2)\ell_1^2\dot{\theta}_1 + m_2\ell_1\ell_2\dot{\theta}_2\cos(\theta_1 - \theta_2) \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) &= (m_1 + m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2\ell_1\ell_2\dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_2 - \dot{\theta}_1) \\
\frac{\partial L}{\partial q_1} &= -m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - (m_1 + m_2)g\ell_1\sin(\theta_1) \\
\frac{\partial L}{\partial \dot{q}_2} &= m_2\ell_2^2\dot{\theta}_2 + m_2\ell_1\ell_2\dot{\theta}_1\cos(\theta_1 - \theta_2) \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) &= m_2\ell_2^2\ddot{\theta}_2 + m_2\ell_1\ell_2\ddot{\theta}_1\cos(\theta_1 - \theta_2) + m_2\ell_1\ell_2\dot{\theta}_1\sin(\theta_1 - \theta_2)(\dot{\theta}_2 - \dot{\theta}_1) \\
\frac{\partial L}{\partial q_2} &= m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - m_2 g\ell_2\sin(\theta_2).
\end{aligned}
$$

If we input these into (6) we will have

$$\begin{aligned}\tau_1 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} \\ &= (m_1+m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2\ell_1\ell_2\dot{\theta}_2\sin(\theta_1-\theta_2)(\dot{\theta}_2-\dot{\theta}_1) \\ &\quad + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + (m_1+m_2)g\ell_1\sin(\theta_1) \\ &= (m_1+m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2\ell_1\ell_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) + (m_1+m_2)g\ell_1\sin(\theta_1)\end{aligned}$$

and

$$\begin{aligned}\tau_2 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} \\ &= m_2\ell_2^2\ddot{\theta}_2 + m_2\ell_1\ell_2\ddot{\theta}_1\cos(\theta_1-\theta_2) + m_2\ell_1\ell_2\dot{\theta}_1\sin(\theta_1-\theta_2)(\dot{\theta}_2-\dot{\theta}_1) \\ &\quad - m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + m_2g\ell_2\sin(\theta_2) \\ &= m_2\ell_2^2\ddot{\theta}_2 + m_2\ell_1\ell_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2\ell_1\ell_2\dot{\theta}_1^2\sin(\theta_1-\theta_2) + m_2g\ell_2\sin(\theta_2).\end{aligned}$$

The special case where $\tau_1 = \tau_2 = 0$ can be found in the appendix of [1]. In state space form we have

$$\begin{aligned}\begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix} &= \begin{bmatrix}(m_1+m_2)\ell_1^2 & m_2\ell_1\ell_2\cos(\theta_1-\theta_2) \\ m_2\ell_1\ell_2\cos(\theta_1-\theta_2) & m_2\ell_2^2\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix}(m_1+m_2)g\ell_1\sin(\theta_1) \\ m_2g\ell_2\sin(\theta_2)\end{bmatrix} \\ &\quad + \begin{bmatrix}0 & m_2\ell_1\ell_2\dot{\theta}_2\sin(\theta_1-\theta_2) \\ -m_2\ell_1\ell_2\dot{\theta}_1\sin(\theta_1-\theta_2) & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix}\end{aligned} \tag{7}$$

which can be succinctly written as

$$\tau = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q). \tag{8}$$

To transform this into a more familiar form, we can have

$$\begin{bmatrix}\dot{q} \\ \ddot{q}\end{bmatrix} = \begin{bmatrix}\dot{q} \\ -M^{-1}(q)[C(q,\dot{q})\dot{q} + G(q)]\end{bmatrix} + \begin{bmatrix}0 \\ M^{-1}(q)\end{bmatrix}\tau \tag{9}$$

which is in the nonlinear control affine form $\dot{x} = f(x) + g(x)u$.

2 MIMO Input-Output Feedback Linearization

In this section, we use the output

$$y = h(x) = \begin{bmatrix}y_1 \\ y_2\end{bmatrix} = \begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix} \tag{10}$$

and state

$$x = \begin{bmatrix}x_1 \\ x_2 \\ x_3 \\ x_4\end{bmatrix} = \begin{bmatrix}\theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix} \tag{11}$$

2.1 Nonlinear Change of Coordinates

And we want to find a nonlinear change of coordinates $z = \Phi(x)$ that renders the input-output relationship linear. First we set $z_{11} = y_1 = \theta_1$, and by getting its higher-order derivatives we have

$$\begin{aligned}\dot{z}_{11} &= \dot{\theta}_1 = x_3 = z_{12} \\ \dot{z}_{12} &= \ddot{\theta}_1 = -\Big[M^{-1}(q)[C(q,\dot{q})\dot{q} + G(q)]\Big]_1 + \Big[M^{-1}(q)\Big]_1\tau = v_1\end{aligned}$$

similarly we define $z_{21} = y_2 = \theta_2$ and we have

$$\begin{aligned}\dot{z}_{21} &= \dot{\theta}_2 = x_4 = z_{22} \\ \dot{z}_{22} &= \ddot{\theta}_2 = -\Big[M^{-1}(q)[C(q,\dot{q})\dot{q} + G(q)]\Big]_2 + \Big[M^{-1}(q)\Big]_2\tau = v_2.\end{aligned}$$

We can see that in this case the vector relative degree is $\{2, 2\}$, thus, we can say the system has relative degree $n = 4$. In state space form we have

$$\begin{bmatrix} \dot{z}_{11} \\ \dot{z}_{21} \\ \dot{z}_{12} \\ \dot{z}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_{11} \\ z_{21} \\ z_{12} \\ z_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{12}$$

which can be written as

$$\dot{z} = Az + Bv. \tag{13}$$

The new control input is $v = [v_1,\ v_2]^T$, which has the following relationship with the true control input τ

$$\tau = Mv + C(q, \dot{q})\dot{q} + G(q). \tag{14}$$

We can define the change of coordinates $\Phi(x)$ as

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \Phi(x) = x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z_{11} \\ z_{21} \\ z_{12} \\ z_{22} \end{bmatrix}. \tag{15}$$

2.2 Discrete Time System

Now we have the system in the form of (13), however, it is in continuous time formulation, since our controller can only change control actions at a fixed rate we need to transform this system into its discrete time counterpart. For a general linear time-invariant (LTI) system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{16}$$

we can get $x(t)$ the state at time t as

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{17}$$

If at time t_k we have $x(t_k)$ and we fix the control $u(t)$ at u_k for a time interval Δt, and we define $t_{k+1} = t_k + \Delta t$ we can have

$$\begin{aligned} x(t_{k+1}) &= e^{A\Delta t}x(t_k) + \int_0^{\Delta t} e^{A(\Delta t-\tau)}Bu_k d\tau \\ &= \underbrace{e^{A\Delta t}}_{\bar{A}}x(t_k) + \underbrace{e^{A\Delta t}\int_0^{\Delta t} e^{-A\tau}d\tau B}_{\bar{B}}\, u_k \end{aligned}$$

which gives us

$$x(t_{k+1}) = \bar{A}x(t_k) + \bar{B}u_k. \tag{18}$$

First we can calculate $\bar{A} = e^{A\Delta t}$ using the matrices obtained in (12). From Taylor's expansion we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{19}$$

thus we have

$$e^{A\Delta t} = I + A\Delta t + \frac{\Delta t^2}{2!}A^2 + \frac{\Delta t^3}{3!}A^3 + \cdots. \tag{20}$$

To find a pattern we can write out the first few A^n's

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

thus we have

$$\bar{A} = e^{A\Delta t} = I + A\Delta t = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{21}$$

Then let's calculate $\bar{B}$, the only unknown component is the integral, which can be calculated as

$$\int_0^{\Delta t} e^{-A\tau} d\tau = \int_0^{\Delta t} \begin{bmatrix} 1 & 0 & -\tau & 0 \\ 0 & 1 & 0 & -\tau \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} \int_0^{\Delta t} 1 d\tau & 0 & -\int_0^{\Delta t} \tau d\tau & 0 \\ 0 & \int_0^{\Delta t} 1 d\tau & 0 & -\int_0^{\Delta t} \tau d\tau \\ 0 & 0 & \int_0^{\Delta t} 1 d\tau & 0 \\ 0 & 0 & 0 & \int_0^{\Delta t} 1 d\tau \end{bmatrix}$$

$$= \begin{bmatrix} \Delta t & 0 & -1/2\Delta t^2 & 0 \\ 0 & \Delta t & 0 & -1/2\Delta t^2 \\ 0 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & \Delta t \end{bmatrix}$$

which inputting into the $\bar{B}$ calculation gives us

$$\bar{B} = \begin{bmatrix} 1/2\Delta t^2 & 0 \\ 0 & 1/2\Delta t^2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix}. \tag{22}$$

This gives us the discrete time version of the system defined in (12)

$$z_{k+1} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} z_k + \begin{bmatrix} 1/2\Delta t^2 & 0 \\ 0 & 1/2\Delta t^2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} v_k, \tag{23}$$

which succinctly we can write as

$$z_{k+1} = \bar{A} z_k + \bar{B} v_k \tag{24}$$

2.3 LQR Target Reaching and Reference Tracking

Then for this linear system we can use LQR to achieve target reaching and reference tracking. We first define the control problem as

$$\min_{\{u_i\}_{i=1,\cdots,N}} \frac{1}{2}(x_N - \bar{x}_N)^T Q_N (x_N - \bar{x}_N) + \sum_{n=0}^{N} \frac{1}{2}(x_n - \bar{x}_n)^T Q_n (x_n - \bar{x}_n) + u_n^T R_n u_n \tag{25}$$

$$\text{subject to } x_{n+1} = A_n x_n + B_n u_n + \omega_n$$

and the optimal LQR controller has the form of $u_n = K_n x_n + k_n$, where

$$K_n = -(R_n + B_n^T P_{n+1} B_n)^{-1} B_n^T P_{n+1} A_n \tag{26a}$$

$$P_n = Q_n + A_n^T P_{n+1} A_n + A_n^T P_{n+1} B_n K_n \tag{26b}$$

$$k_n = -(R_n + B_n^T P_{n+1} B_n)^{-1} B_n^T p_{n+1} \tag{26c}$$

$$p_n = q_n + A_n^T p_{n+1} + A_n^T P_{n+1} B_n k_n \tag{26d}$$

$$q_n = -Q_n \bar{x}_n \tag{26e}$$

here the sequence is initialized as $P_N = Q_N$ and $p_N = q_N$. Using LQR on the system defined in (23) we can then move the state to the target state or make it follow a reference trajectory. If we are simply trying to reach a non-zero target we can solve the discrete-time algebraic Ricatti equation

$$P = Q + A^T PA - A^T PB(B^T PB + R)^{-1} B^T PA \tag{27a}$$

$$K = -(B^T PB + R)^{-1} B^T PA \tag{27b}$$

and we get the control as

$$u = K(x - \tilde{x}) \tag{28}$$

where $\tilde{x}$ is the non-zero target. The overall procedure is given in Figure 2.

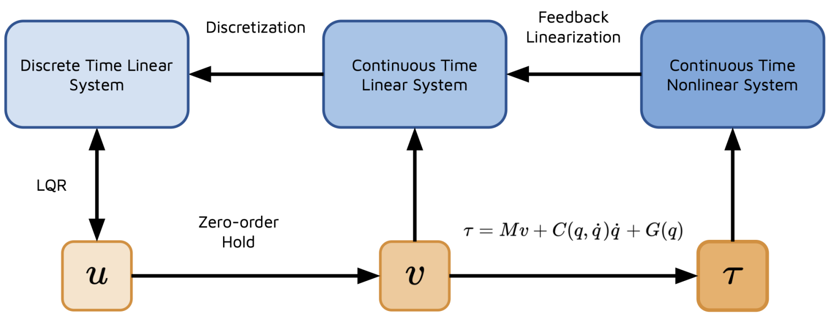

Figure 2: **Feedback Linearization Procedure.**

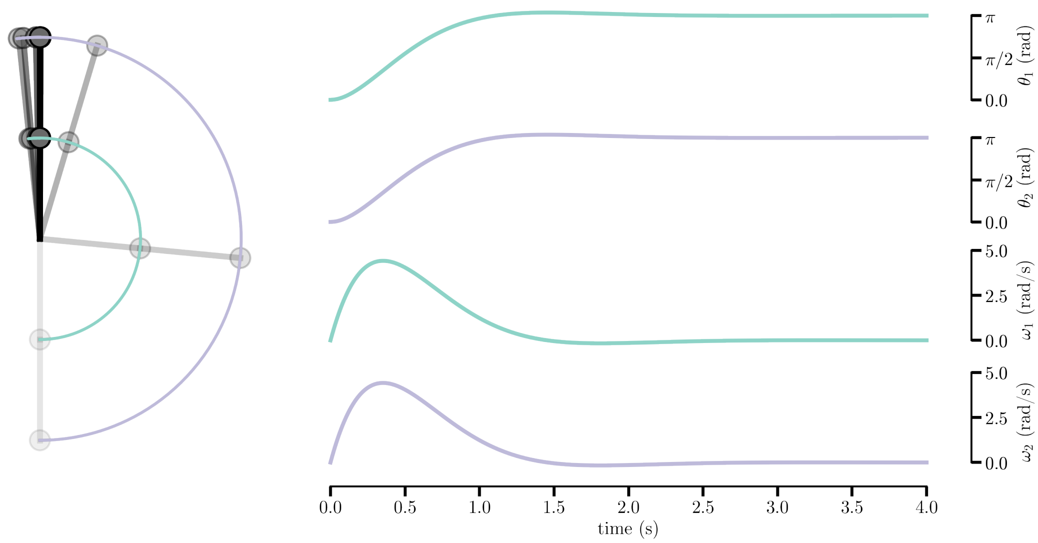

Figure 3: **Swing-Up Trajectory**. The illustration on the left shows snapshots of double pendulum along the trajectory. The snapshots are taken evenly spaced in time. We can see that that once the double pendulum is near the top it oscillates for a relative long period compared to the swing-up phase. The four plots on the right shows the angle and angular velocity of the two links over time.

3 Experimental Results

In this section, we will show the experimental results for a double pendulum swing-up task and a reference trajectory tracking task.

3.1 Swing-Up Task

In this section we will show how to create a swing-up controller for a double pendulum model. The nonlinear model is defined in (9) and the feedback linearized model is defined in (12). The first step is to discretize the linear model, which gives us the model in (23). Then we can use LQR to find the controller that moves the discrete time linear model to the top, we can either use the reference tracking version in (26) or the stabilizing version in (27). Here the initial state is $[0,\ 0,\ 0,\ 0]^T$ and the target state is $[\pi,\ \pi,\ 0,\ 0]^T$. Given the way we discretized the system, this discrete time controller will also work on the continuous time model. Once we have the controller v for the linear system we can calculate the corresponding controller for the nonlinear system τ using (14). The performance of the resulting controller is shown in Figure 3. We can see that the double pendulum system behaves like a single pendulum system, since the two links are always kept on the same line.

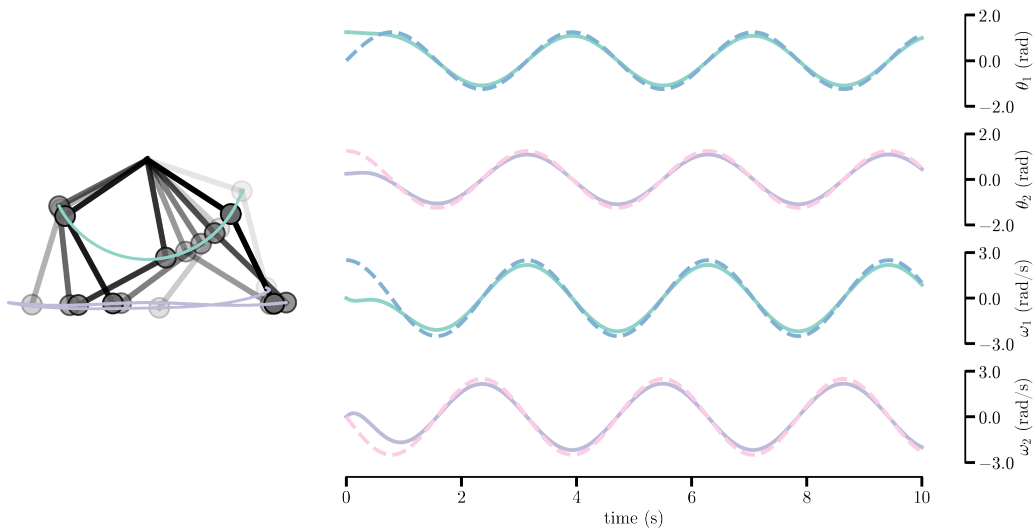

Figure 4: **Reference Tracking**. The illustration on the left shows snapshots of double pendulum along the trajectory. The snapshots are taken evenly spaced in time. The four plots on the right shows the angle and angular velocity of the two links over time, the solid line denotes the actual trajectory and the dashed line denotes the reference signal.

3.2 Reference Tracking

In this part of this report, we show the results of the feedback linearization controller on a reference tracking task. To be specific, the reference signal that is tracked is

$$\theta_1 = 1.25 \sin(2t) \tag{29a}$$
$$\theta_2 = 1.25 \cos(2t) \tag{29b}$$

which implies that

$$\dot{\theta}_1 = 2.5 \cos(2t) \tag{30a}$$
$$\dot{\theta}_2 = -2.5 \sin(2t). \tag{30b}$$

The controller is implemented similar to the swing-up controller, the main difference is the LQR controller is confined to the design in (26). The performance of the resulting controller is shown in Figure 4. It can be seen, that despite the initial error, the controller made the system follow the reference trajectory in less than 2 seconds. The initial state is

$$\begin{bmatrix} \theta_1(0) \\ \theta_2(0) \\ \dot{\theta}_1(0) \\ \dot{\theta}_2(0) \end{bmatrix} = \begin{bmatrix} 1.25 \\ 0.25 \\ 0.00 \\ 0.00 \end{bmatrix}.$$

References

[1] T. Shinbrot, C. Grebogi, J. Wisdom, and J. A. Yorke. Chaos in a double pendulum. *American Journal of Physics*, 60(6):491–499, 1992.