Walking motion generation Notes

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1 Introduction

Running a biped robot is achieved by solving a series of optimization problems during execution. For simple cases this can be realized at a relatively high frequency. However, when the optimization problem gets larger, such as for higher order systems or having nonlinear constraints, the solution takes longer to obtain. This impairs the ability for the robot to react to sudden changes in its trajectory, e.g., a push. This work first utilizes learning-based methods in a supervised learning approach to learn the solution of a convex optimizer.

The remainder of this paper is structured as follows: in Section II a commonly used simplified model for bipedal locomotion, namely the linear inverted pendulum model (LIPM) is presented, and simplifications for walking on a flat ground is given; in section III we show how to plan the CoM, CoP and foot step sequences using MPC. The code for what this note is talking about can be found in this GitHub Repository (click me) and a video showing the results can be found on YouTube.

2 Linear Inverted Pendulum Model

First, the derivation of the relationship between the center of mass (CoM) and the center of pressure (CoP) is given. We start from the Newton-Euler equations

$$m\ddot{c} = \sum_i f_i - mg \qquad \text{(Newton's Equation)} \tag{1}$$

$$\dot{L} = \sum_i (p_i - c) \times f_i, \qquad \text{(Euler's Equation)} \tag{2}$$

where the mass is m, the position of the center of mass (CoM) is c, the angular momentum of the CoM is L, p_i is the of where force f_i is applied. The derivation is as follow:

$$\begin{aligned}
mc \times (\ddot{c} + g) &= \sum_i c \times f_i && \text{the cross product of } c \text{ and (1)}\\
mc \times (\ddot{c} + g) + \dot{L} &= \sum_i c \times f_i + \sum_i (p_i - c) \times f_i && \text{add (2) on both sides}\\
mc \times (\ddot{c} + g) + \dot{L} &= \sum_i p_i \times f_i\\
\frac{mc \times (\ddot{c} + g) + \dot{L}}{m(\ddot{c}^z + g^z)} &= \frac{\sum_i p_i \times f_i}{m(\ddot{c}^z + g^z)} && \text{divide both sides with } m(\ddot{c}^z + g^z)\\
\frac{1}{(\ddot{c}^z + g^z)} \begin{bmatrix} c^y(\ddot{c}^z + g) - c^z\ddot{c}^y \\ c^z\ddot{c}^x - c^x(\ddot{c}^z + g) \\ c^x\ddot{c}^y - c^y\ddot{c}^x \end{bmatrix} &= \sum_i \frac{1}{m(\ddot{c}^z + g^z)} \begin{bmatrix} p_i^y f_i^z - p_i^z f_i^y - \dot{L}^x \\ p_i^z f_i^x - p_i^x f_i^z - \dot{L}^y \\ p_i^x f_i^y - p_i^y f_i^x - \dot{L}^z \end{bmatrix}\\
\frac{1}{(\ddot{c}^z + g^z)} \begin{bmatrix} c^y(\ddot{c}^z + g) - c^z\ddot{c}^y \\ c^z\ddot{c}^x - c^x(\ddot{c}^z + g) \\ c^x\ddot{c}^y - c^y\ddot{c}^x \end{bmatrix} &= \sum_i \frac{1}{m(\ddot{c}^z + g^z)} \begin{bmatrix} p_i^y f_i^z - \dot{L}^x \\ -p_i^x f_i^z - \dot{L}^y \\ p_i^x f_i^y - p_i^y f_i^x - \dot{L}^z \end{bmatrix}. && f_i\text{'s are applied on the ground } (p_i^z = 0)
\end{aligned}$$

This gives us

$$\frac{1}{(\ddot{c}^z + g^z)} \begin{bmatrix} c^y(\ddot{c}^z + g) - c^z\ddot{c}^y \\ c^z\ddot{c}^x - c^x(\ddot{c}^z + g) \\ c^x\ddot{c}^y - c^y\ddot{c}^x \end{bmatrix} = \sum_i \frac{1}{\sum_i f_i^z} \begin{bmatrix} p_i^y f_i^z \\ -p_i^x f_i^z \\ p_i^x f_i^y - p_i^y f_i^x \end{bmatrix} + \frac{1}{m(\ddot{c}^z + g^z)} \begin{bmatrix} \dot{L}^x \\ \dot{L}^y \\ \dot{L}^z \end{bmatrix},$$

which rewrites $m(\ddot{c}^z + g^z)$ as $\sum_i f_i^z$. If we only look at the x and y components of the above equation we have

$$\begin{cases} c^x - \dfrac{c^z}{(\ddot{c}^z + g^z)}\ddot{c}^y + \dfrac{1}{m(\ddot{c}^z + g^z)}\dot{L}^y = \dfrac{\sum_i p_i^x f_i^z}{\sum_i f_i^z} = p^x \\ c^y - \dfrac{c^z}{(\ddot{c}^z + g^z)}\ddot{c}^y - \dfrac{1}{m(\ddot{c}^z + g^z)}\dot{L}^x = \dfrac{\sum_i p_i^y f_i^z}{\sum_i f_i^z} = p^y \end{cases}$$

Which in matrix form is

$$\mathbf{c}^{x,y} - \frac{c^z}{(\ddot{c}^z + g^z)}\ddot{\mathbf{c}}^{x,y} + \frac{1}{m(\ddot{c}^z + g^z)}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\dot{\mathbf{L}}^{x,y} = \mathbf{p}^{x,y} \tag{3}$$

where

$$\mathbf{c}^{x,y} = \begin{bmatrix} c^x \\ c^y \end{bmatrix}, \mathbf{L}^{x,y} = \begin{bmatrix} L^x \\ L^y \end{bmatrix}, \mathbf{p}^{x,y} = \begin{bmatrix} p^x \\ p^y \end{bmatrix}.$$

Under the assumptions

$$\begin{aligned} \ddot{c}^z &= 0 && \text{not jumping} \\ \dot{\mathbf{L}}^{x,y} &= 0, && \text{no huge rotations} \end{aligned}$$

we can write (3) as

$$\mathbf{c}^{x,y} - \frac{c^z}{g^z}\ddot{\mathbf{c}}^{x,y} = \mathbf{p}^{x,y}. \tag{4}$$

This is know as the linearly inverted pendulum model (LIPM). If we treat the CoP location $\mathbf{p}^{x,y}$ as the control input we can have the dynamical system as

$$\ddot{\mathbf{c}}^{x,y} = \frac{g^z}{c^z}\mathbf{c}^{x,y} - \frac{g^z}{c^z}\mathbf{p}^{x,y}. \tag{5}$$

Equation (5) can be written as

$$\begin{bmatrix} \dot{c}^x \\ \dot{c}^y \\ \ddot{c}^x \\ \ddot{c}^y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ g^z/c^z & 0 & 0 & 0 \\ 0 & g^z/c^z & 0 & 0 \end{bmatrix} \begin{bmatrix} c^x \\ c^y \\ \dot{c}^x \\ \dot{c}^y \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ g^z/c^z & 0 \\ 0 & g^z/c^z \end{bmatrix} \begin{bmatrix} \dot{p}^x \\ \dot{p}^y \end{bmatrix}. \tag{6}$$

3 Biped Walking Problem

3.1 Motion of the Center of Mass

For the biped walking we use a dynamic model where the input is jerk. The relationship between position, velocity, acceleration and jerk is

$$J(t) = J \tag{7a}$$
$$a(t) = Jt + a_0 \tag{7b}$$
$$v(t) = \frac{1}{2}Jt^2 + a_0 t + v_0 \tag{7c}$$
$$x(t) = \frac{1}{6}Jt^3 + \frac{1}{2}a_0 t^2 + v_0 t + x_0. \tag{7d}$$

Thus, if we set the state to be

$$\hat{x}_k = \begin{bmatrix} x(t_k) \\ \dot{x}(t_k) \\ \ddot{x}(t_k) \end{bmatrix}, \tag{8}$$

then we can write the dynamic system as

$$\begin{bmatrix} x(t_{k+1}) \\ \dot{x}(t_{k+1}) \\ \ddot{x}(t_{k+1}) \end{bmatrix} = \begin{bmatrix} \frac{1}{6}Jt^3 + \frac{1}{2}a_0 t^2 + v_0 t + x_0 \\ \frac{1}{2}Jt^2 + a_0 t + v_0 \\ Jt + a_0 \end{bmatrix}.$$

Since we can write J as $\dddot{x}$ then we have a linear system $\hat{x}_{k+1} = A\hat{x}_k + B\dddot{x}_k$,

$$\begin{bmatrix} x(t_{k+1}) \\ \dot{x}(t_{k+1}) \\ \ddot{x}(t_{k+1}) \end{bmatrix} = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t_k) \\ \dot{x}(t_k) \\ \ddot{x}(t_k) \end{bmatrix} + \begin{bmatrix} t^3/6 \\ t^2/2 \\ t \end{bmatrix} \dddot{x}, \tag{9}$$

the same goes for y. Using the new state we can rewrite (4) as

$$\mathbf{p}_k^x = \begin{bmatrix} 1 & 0 & -c^z/g^z \end{bmatrix} \hat{x}_k \tag{10a}$$
$$\mathbf{p}_k^y = \begin{bmatrix} 1 & 0 & -c^z/g^z \end{bmatrix} \hat{y}_k. \tag{10b}$$

Now we would like to write the future states $\{\begin{bmatrix} x_{k+i} & \dot{x}_{k+i} & \ddot{x}_{k+i} \end{bmatrix}^T\}_{i=1}^N$ as functions of the current state $\hat{x}_k$ and future jerks $\dddot{x}_j$, $j = k, \cdots, k+N-1$. Starting from

$$\hat{x}_{k+1} = A\hat{x}_k + B\dddot{x}(t_k)$$

we can have the following result,

$$\begin{aligned}
\hat{x}_{k+2} &= A\hat{x}_{k+1} + B\dddot{x}(t_k+1) \\
&= A^2\hat{x}_k + AB\dddot{x}(t_k) + B\dddot{x}(t_k+1) \\
\hat{x}_{k+3} &= A\hat{x}_{k+2} + B\dddot{x}(t_k+2) \\
&= A^3\hat{x}_k + A^2B\dddot{x}(t_k) + AB\dddot{x}(t_k+1) + B\dddot{x}(t_k+2) \\
&\vdots \\
\hat{x}_{k+N} &= A\hat{x}_{k+N-1} + B\dddot{x}(t_{k+N-1}) \\
&= A^N\hat{x}_k + [A^{N-1}B\dddot{x}(t_k) + A^{N-2}B\dddot{x}(t_k+1) + \cdots + B\dddot{x}(t_{k+N-1})].
\end{aligned}$$

We can see that

$$A^n = \begin{bmatrix} 1 & nt & (n^2/2)t^2 \\ 0 & 1 & nt \\ 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

To prove this we can write

$$A = I + tI' + \frac{t^2}{2}I'' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

Using the following properties

$$\begin{aligned}
I'I' &= I'' \\
I''I'' &= \mathbf{0} \\
I'I'' &= I''I' = \mathbf{0},
\end{aligned}$$

we can have

$$\begin{aligned}
A^2 &= (I + tI' + \frac{t^2}{2}I'')(I + tI' + \frac{t^2}{2}I'') \\
&= (I + tI' + \frac{t^2}{2}I'') + (tI' + t^2I'' + 0) + (\frac{t^2}{2}I'' + 0 + 0) \\
&= I + 2tI' + \frac{4}{2}t^2I'' \\
&= \begin{bmatrix} 1 & 2t & (2^2/2)t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

If we have A^k satisfying (11) we can have

$$\begin{aligned} A^{k+1} &= (I + ktI' + \frac{k^2}{2}t^2 I'')(I + tI' + \frac{t^2}{2}I'') \\ &= (I + tI' + \frac{t^2}{2}I'') + (ktI' + kt^2 I'' + 0) + (\frac{k^2}{2}t^2 I'' + 0 + 0) \\ &= I + (k+1)tI' + (\frac{1}{2} + k + \frac{k^2}{2})t^2 I'' \\ &= I + (k+1)tI' + \frac{k^2+2k+1}{2}t^2 I'' \\ &= \begin{bmatrix} 1 & (k+1)t & [(k+1)^2/2]t^2 \\ 0 & 1 & (k+1)t \\ 0 & 0 & 1 \end{bmatrix}. \qquad \square \end{aligned}$$

Thus we can say if A^k satisfies (11), then A^{k+1} also satisfies (11).

We want to write We can have

$$\begin{aligned} \hat{x}_{k+m} &= A^m \hat{x}_k + [A^{m-1}B\dddot{x}(t_k) + A^{m-2}B\dddot{x}(t_k+1) + \cdots + B\dddot{x}(t_{k+m-1})] \\ &= \begin{bmatrix} 1 & mt & (m^2/2)t^2 \\ 0 & 1 & mt \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t_k) \\ \dot{x}(t_k) \\ \ddot{x}(t_k) \end{bmatrix} + \begin{bmatrix} (A^{m-1}B)_1 \\ (A^{m-1}B)_2 \\ (A^{m-1}B)_3 \end{bmatrix} \dddot{x}(t_k) + \cdots + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \dddot{x}(t_m - 1). \end{aligned}$$

Therefore, we have

$$x_{k+m} = \begin{bmatrix} 1 & mt & (m^2/2)t^2 \end{bmatrix} \hat{x}_k + \begin{bmatrix} (A^{m-1}B)_1 & \cdots & B_1 \end{bmatrix} \begin{bmatrix} \dddot{x}(t_k) \\ \vdots \\ \dddot{x}(t_m - 1) \end{bmatrix}$$

$$\dot{x}_{k+m} = \begin{bmatrix} 0 & 1 & mt \end{bmatrix} \hat{x}_k + \begin{bmatrix} (A^{m-1}B)_2 & \cdots & B_2 \end{bmatrix} \begin{bmatrix} \dddot{x}(t_k) \\ \vdots \\ \dddot{x}(t_m - 1) \end{bmatrix}$$

$$\ddot{x}_{k+m} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \hat{x}_k + \begin{bmatrix} (A^{m-1}B)_3 & \cdots & B_3 \end{bmatrix} \begin{bmatrix} \dddot{x}(t_k) \\ \vdots \\ \dddot{x}(t_m - 1) \end{bmatrix}.$$

Therefore we have $S_p, S_v, S_a \in \mathbb{R}^{n\times 3}$,

$$S_p = \begin{bmatrix} 1 & T & T^2/2 \\ \vdots & \vdots & \vdots \\ 1 & nT & (n^2/2)T^2 \end{bmatrix}, S_v = \begin{bmatrix} 0 & 1 & T \\ \vdots & \vdots & \vdots \\ 0 & 1 & nT \end{bmatrix} \text{ and } S_a = \begin{bmatrix} 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{bmatrix},$$

and $U_p, U_v, U_a \in \mathbb{R}^{n\times n}$,

$$U_p = \begin{bmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ (A^{N-1}B)_1 & \cdots & B_1 \end{bmatrix}, U_v = \begin{bmatrix} B_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ (A^{N-1}B)_2 & \cdots & B_2 \end{bmatrix} \text{ and } U_a = \begin{bmatrix} B_3 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ (A^{N-1}B)_3 & \cdots & B_3 \end{bmatrix},$$

with

$$A^k B = \begin{bmatrix} \frac{1}{6}t^3 + \frac{k}{2}t^3 + \frac{k^2}{2}t^3 \\ \frac{1}{2}t^2 + kt^2 \\ t \end{bmatrix} = \begin{bmatrix} (A^k B)_1 \\ (A^k B)_2 \\ (A^k B)_3 \end{bmatrix},$$

and $B = A^0 B$. Using the aforementioned matrices we can have the following relationship

$$X_{k+1} = \begin{bmatrix} x_{k+1} \\ \vdots \\ x_{k+N} \end{bmatrix} = S_p \hat{x}_k + U_p \dddot{X}_k$$

$$\dot{X}_{k+1} = \begin{bmatrix} \dot{x}_{k+1} \\ \vdots \\ \dot{x}_{k+N} \end{bmatrix} = S_v \hat{x}_k + U_v \dddot{X}_k$$

$$\ddot{X}_{k+1} = \begin{bmatrix} \ddot{x}_{k+1} \\ \vdots \\ \ddot{x}_{k+N} \end{bmatrix} = S_a \hat{x}_k + U_a \dddot{X}_k$$

$$\dddot{X}_k = \begin{bmatrix} \dddot{x}_k \\ \vdots \\ \dddot{x}_{k+N-1} \end{bmatrix}.$$

3.2 Motion of the Center of Pressure

Also we can write the relationship between $\hat{x}_k$, $\dddot{X}_k$ and the CoP

$$Z^x_{k+1} = \begin{bmatrix} z^x_{k+1} \\ \vdots \\ z^x_{k+N} \end{bmatrix} = S_z \hat{x}_k + U_z \dddot{X}_k \tag{13}$$

with $S_z \in \mathbb{R}^{n\times 3}$ and $U_z \in \mathbb{R}^{n\times n}$

$$S_z = S_p - \frac{h}{g} S_a \tag{14a}$$

$$U_z = U_p - \frac{h}{g} U_a. \tag{14b}$$

We can see that this makes sense by inputting (14) to (13)

$$\begin{aligned} S_z \hat{x}_k + U_z \dddot{X}_k &= (S_p - \frac{h}{g} S_a)\hat{x}_k + (U_p - \frac{h}{g} U_a)\dddot{X}_k \\ &= S_p \hat{x}_k + U_p \dddot{X}_k - \frac{h}{g} S_a \hat{x}_k - \frac{h}{g} U_a \dddot{X}_k \\ &= X_{k+1} - \frac{h}{g} \ddot{X}_{k+1}, \end{aligned}$$

which is a vector version of (4)

3.3 Motion of the feet on the ground

Instead of pre-defining the foot step locations we would like to optimize it on the fly. To do this we introduce new decision variables to the optimization problem $X^f_k, Y^f_k \in \mathbb{R}^m$ which represents the foot step location of the next m foot steps,

$$X^f_k = \begin{bmatrix} (X^f_k)_1 \\ \vdots \\ (X^f_k)_m \end{bmatrix} \text{ and } Y^f_k = \begin{bmatrix} (Y^f_k)_1 \\ \vdots \\ (Y^f_k)_m \end{bmatrix},$$

where the location the next l-th foot step is $\left((X^f_k)_l, (Y^f_k)_l\right)$. This also provides an adapting reference to the CoP

$$Z^{x_{\text{ref}}}_{k+1} = U^c_{k+1} X^{fc}_k + U_{k+1} X^f_k \tag{15a}$$

$$Z^{y_{\text{ref}}}_{k+1} = U^c_{k+1} Y^{fc}_k + U_{k+1} Y^f_k. \tag{15b}$$

Define $\mathbf{1}_m$ and $\mathbf{0}_m$ which denotes column vectors of 1 and 0 with dimension m, respectively. We have

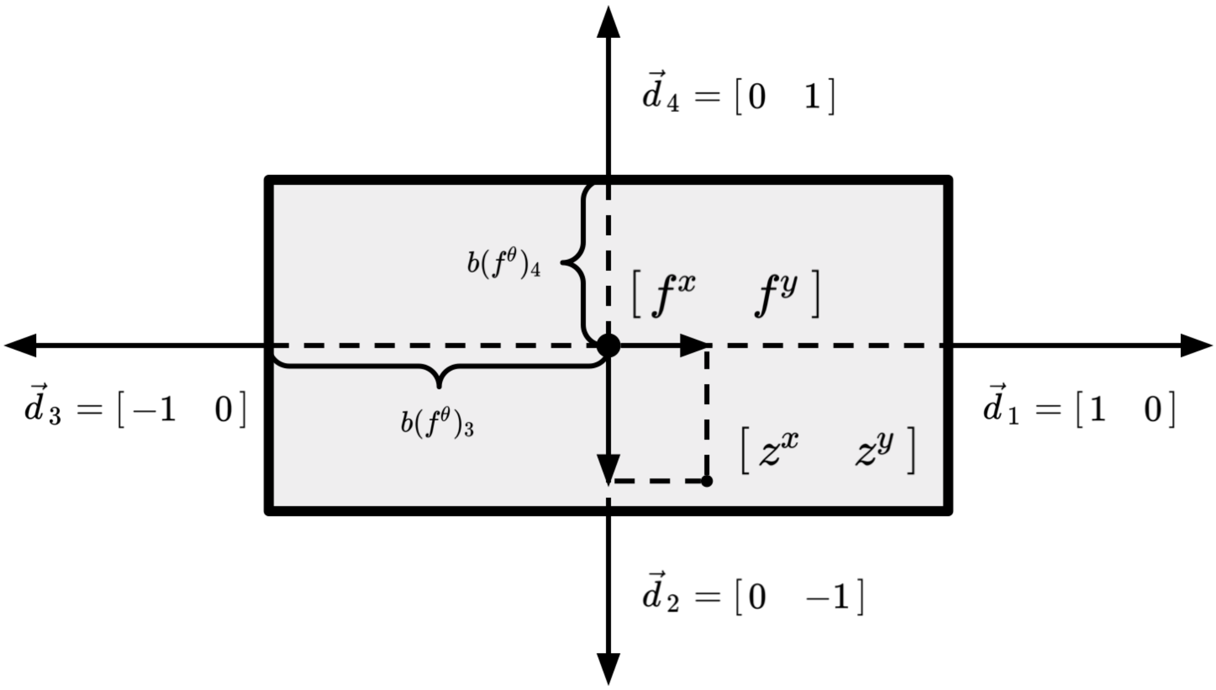

Figure 1: **Illustration of CoP Constraint.**

$$U^c_{k+1} = \begin{bmatrix} \mathbf{1}_m \\ \mathbf{0}_M \\ \vdots \\ \mathbf{0}_{N-m-lm} \end{bmatrix} \in \mathbb{R}^N \text{ and } U_{k+1} = \begin{bmatrix} \mathbf{0}_m & \cdots & \mathbf{0}_m \\ \mathbf{1}_M & \cdots & \mathbf{0}_M \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{N-m-lm} & \cdots & \mathbf{1}_{N-m-lm} \end{bmatrix} \in \mathbb{R}^{N\times l}$$

where in the preview horizon N we can look into the future l steps where each step has M time steps, the remaining time steps in the current foot step is m.

3.4 Constraints on the Center of Pressure

Because of trajectories with CoP outside of the support polygon cannot be realized we would need to constraint it to be inside. One way to write this is with $s \in \{\text{left}, \text{right}\}$

$$\begin{bmatrix} d^x_s(f^\theta) & d^y_s(f^\theta) \end{bmatrix} \begin{bmatrix} z^x - f^x \\ z^y - f^y \end{bmatrix} \leq b(f^\theta). \tag{16}$$

The column vectors $d^x_s(f^\theta)$ and $d^y_s(f^\theta)$ gather the x and y coordinates the normal vector to the edges of the feet. As illustrated in Figure 1 if we have a rectangle shaped feet its four edges has the normal vectors

$$\vec{d_1} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \vec{d_2} = \begin{bmatrix} 0 & -1 \end{bmatrix}, \vec{d_3} = \begin{bmatrix} -1 & 0 \end{bmatrix} \text{ and } \vec{d_4} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The vector $b(f^\theta) \in \mathbb{R}^e$, where e denotes the number of edges, records the perpendicular distance between the center of the foot $[f^x\ f^y]$ and the edges. If we take the inner product of the normal vector of the i-th edge and the vector between the center of the foot and the CoP position $[z^x\ z^y]$ we can get it projection on the normal vector direction

$$\left(\vec{d_i}\right)^T \begin{bmatrix} z^x - f^x \\ z^y - f^y \end{bmatrix} = \|\vec{d_i}\| \|\vec{z}_{x,y}\| \cos\phi = \|\vec{z}_{x,y}\| \cos\phi,$$

where $\|\vec{d_i}\| = 1$, $\|\vec{z}_{x,y}\| = [(z^x - f^x),\ (z^y - f^y)]^T$, and ϕ is the angle between $\|\vec{z}_{x,y}\|$ and $\|\vec{d_i}\|$. If the length of this projection is less the perpendicular distance to the edge then the CoP is definitely within the support polygon. The θ takes the orientation of the support polygon into consideration, for example is we take the rectangular foot in Figure 1 and rotate it 45° clockwise the normal vectors will also be rotated 45° clockwise. In matrix form we have

$$D_{k+1} \begin{bmatrix} Z^x_{k+1} - U^c_{k+1} X^{fc}_k - U_{k+1} X^f_k \\ Z^y_{k+1} - U^c_{k+1} Y^{fc}_k - U_{k+1} Y^f_k \end{bmatrix} \leq b_{k+1}, \tag{17}$$

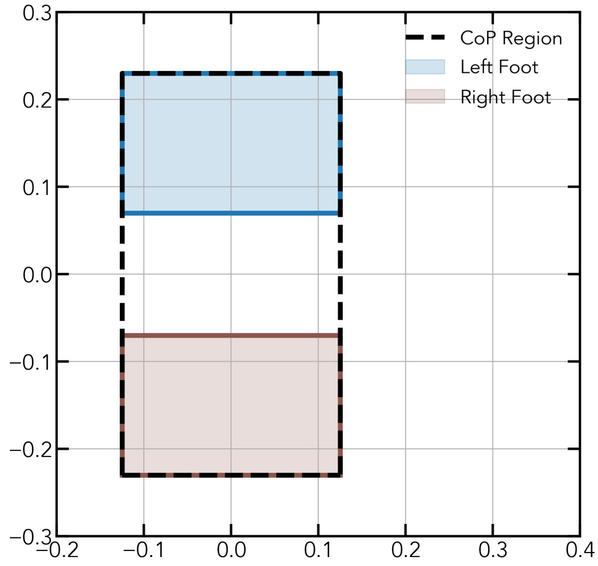

Figure 2: **CoP constraint for the initial double support phase.**

with $D_{k+1} \in \mathbb{R}^{eN\times 2N}$ and $b_{k+1} \in \mathbb{R}^{eN\times 1}$

$$D_{k+1} = \begin{bmatrix} d^x_{s_{i+1}}(f^\theta_{i+1}) & \cdots & 0 & d^y_{s_{i+1}}(f^\theta_{i+1}) & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & d^x_{s_{i+N}}(f^\theta_{i+N}) & 0 & \cdots & d^y_{s_{i+N}}(f^\theta_{i+N}) \end{bmatrix} \tag{18a}$$

$$b_{k+1} = \begin{bmatrix} b(f^\theta_{i+1}) \\ \vdots \\ b(f^\theta_{i+N}) \end{bmatrix}. \tag{18b}$$

Using (15) we can write terms in (17) as

$$\begin{aligned} Z^x_{k+1} - U^c_{k+1}X^{fc}_k - U_{k+1}X^f_k &= S_z\hat{x}_k + U_z\dddot{X}_k - U^c_{k+1}X^{fc}_k - U_{k+1}X^f_k \\ &= \begin{bmatrix} U_z \\ -U_{k+1} \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \dddot{X}_k \\ X^f_k \\ \dddot{Y}_k \\ Y^f_k \end{bmatrix} + S_z\hat{x}_k - U^c_{k+1}X^{fc}_k. \end{aligned}$$

Thus we can write (17) as

$$D_{k+1}\begin{bmatrix} U_z & -U_{k+1} & 0 & 0 \\ 0 & 0 & U_z & -U_{k+1} \end{bmatrix}\begin{bmatrix} \dddot{X}_k \\ X^f_k \\ \dddot{Y}_k \\ Y^f_k \end{bmatrix} \le b_{k+1} + D_{k+1}\begin{bmatrix} U^c_{k+1}X^{fc}_k - S_z\hat{x}_k \\ U^c_{k+1}Y^{fc}_k - S_z\hat{y}_k \end{bmatrix}. \tag{19}$$

3.5 Constraints on the Center of Pressure for Initial Double Support Phase

Starting from (17) we have

$$D_{k+1}\begin{bmatrix} Z^x_{k+1} - U^c_{k+1}X^{fc}_0 - U_{k+1}\texttt{[:, :1]}X^{fc}_1 - U_{k+1}\texttt{[:, 1:]}X^f_1 \\ Z^y_{k+1} - U^c_{k+1}Y^{fc}_0 - U_{k+1}\texttt{[:, :1]}Y^{fc}_1 - U_{k+1}\texttt{[:, 1:]}Y^f_1 \end{bmatrix} \le b_{k+1}. \tag{20}$$

To understand (20), we need to first figure out where the center of the support polygon is for the initial double support phase and the subsequent single support phase. For the initial double support phase, as shown in Fig. 2, the center of the support polygon is the mid-point of the two feet (X^{fc}_0, Y^{fc}_0), which can be set to $(0, 0)$. If first lifting the right foot, then the subsequent single support phase will have the center of the support polygon at the center of the left foot (X^{fc}_1, Y^{fc}_1). The remaining foot step positions are subject to the optimization algorithm, therefore, X^f_1 and Y^f_1 are part of the decision variable. The terms in the matrix can be written as

$$
\begin{aligned}
&Z^x_{k+1} - U^c_{k+1} X^{fc}_0 - U_{k+1}\texttt{[:, :1]} X^{fc}_1 - U_{k+1}\texttt{[:, 1:]} X^f_1 \\
&= S_z \hat{x}_k + U_z \dddot{X}_k - U^c_{k+1} X^{fc}_0 - U_{k+1}\texttt{[:, :1]} X^{fc}_1 - U_{k+1}\texttt{[:, 1:]} X^f_1 \\
&= U_z \dddot{X}_k - U_{k+1}\texttt{[:, 1:]} X^f_1 + S_z \hat{x}_k - U^c_{k+1} X^{fc}_0 - U_{k+1}\texttt{[:, :1]} X^{fc}_1 \\
&= \begin{bmatrix} U_z \\ -U_{k+1}\texttt{[:, 1:]} \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \dddot{X}_k \\ X^f_1 \\ \dddot{Y}_k \\ Y^f_1 \end{bmatrix} + S_z \hat{x}_k - U^c_{k+1} X^{fc}_0 - U_{k+1}\texttt{[:, :1]} X^{fc}_1 .
\end{aligned}
$$

Thus we can write (20) as

$$
\begin{aligned}
&D_{k+1} \begin{bmatrix} U_z & -U_{k+1}\texttt{[:, 1:]} & 0 & 0 \\ 0 & 0 & U_z & -U_{k+1}\texttt{[:, 1:]} \end{bmatrix} \begin{bmatrix} \dddot{X}_k \\ X^f_1 \\ \dddot{Y}_k \\ Y^f_1 \end{bmatrix} \\
&\leq b_{k+1} + D_{k+1} \begin{bmatrix} U^c_{k+1} X^{fc}_0 + U_{k+1}\texttt{[:, :1]} X^{fc}_1 - S_z \hat{x}_k \\ U^c_{k+1} Y^{fc}_0 + U_{k+1}\texttt{[:, :1]} Y^{fc}_1 - S_z \hat{y}_k \end{bmatrix} \\
&\leq b_{k+1} + D_{k+1} \begin{bmatrix} U_{k+1}\texttt{[:, :1]} X^{fc}_1 - S_z \hat{x}_k \\ U_{k+1}\texttt{[:, :1]} Y^{fc}_1 - S_z \hat{y}_k \end{bmatrix}, \qquad \text{assuming } (X^{fc}_0, Y^{fc}_0) = (0, 0)
\end{aligned} \tag{21}
$$

where

$$
\tilde{D}^\alpha_{k+1} = \begin{bmatrix} \tilde{d}^\alpha_{s_{i+1}}(f^\theta_{i+1}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{d}^\alpha_{s_{i+m}}(f^\theta_{i+m}) \end{bmatrix} \tag{22}
$$

$$
D^\alpha_{k+1} = \begin{bmatrix} d^\alpha_{s_{i+m+1}}(f^\theta_{i+m+1}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d^\alpha_{s_{i+N}}(f^\theta_{i+N}) \end{bmatrix} \tag{23}
$$

$$
D_{k+1} = \begin{bmatrix} \tilde{D}^x_{k+1} & \mathbf{0} & \tilde{D}^y_{k+1} & \mathbf{0} \\ \mathbf{0} & D^x_{k+1} & \mathbf{0} & D^y_{k+1} \end{bmatrix} \tag{24}
$$

$$
b_{k+1} = \begin{bmatrix} \tilde{b}(f^\theta_{i+1}) \\ \vdots \\ \tilde{b}(f^\theta_{i+m}) \\ b(f^\theta_{i+m+1}) \\ \vdots \\ b(f^\theta_{i+N}) \end{bmatrix}, \tag{25}
$$

with $\tilde{d}^\alpha_{s_{i+1}}(f^\theta_{i+1})$ denoting the normal vectors of the edges for the initial double support support polygon, $d^\alpha_{s_{i+m+1}}(f^\theta_{i+m+1})$ denoting the normal vectors of the edges for the single support support polygon, $\tilde{b}(f^\theta_{i+1})$ denotes the maximum distance between the CoP and the center of the support polygon along each of the normal vectors for the initial double support support polygon and $b(f^\theta_{i+m+1})$ is its counterpart for the subsequent single support phase.

3.6 Constraints on Foot Step Placements

As mentioned in [1] and [2] the foot step constraint has two parts. The first part is to limit the next footstep position within a convex hull of the position of the current foot step. The second part is limiting the next footstep within a region with respect to the foot in the air. One thing about the first part of the constraint is in both [1] and [2], a clear explanation was not provided. However, in [3] the authors gave a understandable explanation, therefore the following constraints will be derived using their interpretation.

3.6.1 Foot Placement w.r.t. Support Foot

One thing to note is that no matter the support foot being the left foot or the right foot, the feasible positions for the next foot (the convex hulls) are symmetric and the normal vectors to the edges of the convex hull are also symmetric

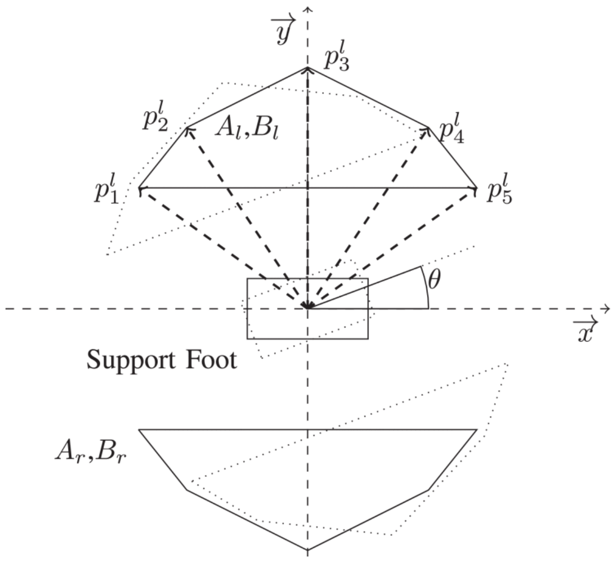

Figure 3: **Illustration of foot placement constraint.**

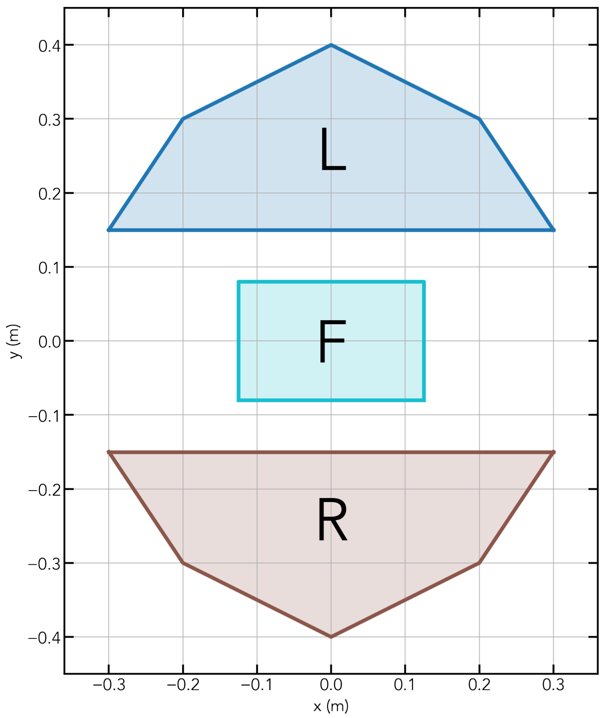

Figure 4: **Constraint w.r.t. support foot**

$$p_i^l = -p_i^r,$$

note using this definition the convex hull of the right foot is not obtained by flipping the convex hull of the left foot along the x-axis, but is obtained by rotating the convex hull of the left foot with respect to the origin clockwise 180°. Here the subscript $i = 1, 2, 3, 4, 5$, denoting the i-th edge of the feasible region. The superscript $\{l, r\}$ denote the left and right foot.

Similar to (16), the constraints are expressed in a form where the projection of the next foot step position on all of the normal vector of the feasible region edges are less than the perpendicular distance between the edges and the current support foot center position. For the convex hull defined in Figure 3 the foot placement constraint has the form of

$$\begin{bmatrix} (p_i^l)_{j+1}^x & (p_i^l)_{j+1}^y \end{bmatrix} \begin{bmatrix} f_{j+1}^x - f_j^x \\ f_{j+1}^y - f_j^y \end{bmatrix} \leq (b_i^l)_{j+1}, \tag{26}$$

where the k denotes the k-th time step, the k-th time step falls within the j-th foot step, therefore the

Table 1: **Edges & Normal Vectors**

Foot	Edge 1	Edge 2	Line Vector	Normal Vector	Distance
Left	$(-0.30, 0.15)$	$(-0.20, 0.30)$	$(2/\sqrt{13}, 3/\sqrt{13})$	$(-3/\sqrt{13}, 2/\sqrt{13})$	$6/5\sqrt{13}$
	$(-0.20, 0.30)$	$(0.00, 0.40)$	$(2/\sqrt{5}, 1/\sqrt{5})$	$(-1/\sqrt{5}, 2/\sqrt{5})$	$4/5\sqrt{5}$
	$(0.30, 0.15)$	$(-0.30, 0.15)$	$(1, 0)$	$(0, -1)$	-0.15
	$(0.00, 0.40)$	$(0.20, 0.30)$	$(2/\sqrt{5}, -1/\sqrt{5})$	$(1/\sqrt{5}, 2/\sqrt{5})$	$4/5\sqrt{5}$
	$(0.20, 0.30)$	$(0.30, 0.15)$	$(2/\sqrt{13}, -3/\sqrt{13})$	$(3/\sqrt{13}, 2/\sqrt{13})$	$6/5\sqrt{13}$
Right	$(-0.30, -0.15)$	$(-0.20, -0.30)$	$(2/\sqrt{13}, -3/\sqrt{13})$	$(-3/\sqrt{13}, -2/\sqrt{13})$	$6/5\sqrt{13}$
	$(-0.20, -0.30)$	$(0.00, -0.40)$	$(2/\sqrt{5}, -1/\sqrt{5})$	$(-1/\sqrt{5}, -2/\sqrt{5})$	$4/5\sqrt{5}$
	$(0.30, -0.15)$	$(-0.30, -0.15)$	$(-1, 0)$	$(0, 1)$	-0.15
	$(0.00, -0.40)$	$(0.20, -0.30)$	$(2/\sqrt{5}, 1/\sqrt{5})$	$(1/\sqrt{5}, -2/\sqrt{5})$	$4/5\sqrt{5}$
	$(0.20, -0.30)$	$(0.30, -0.15)$	$(2/\sqrt{13}, 3/\sqrt{13})$	$(3/\sqrt{13}, -2/\sqrt{13})$	$6/5\sqrt{13}$

subscripts for the edges and perpendicular distances are w.r.t. to the j-th foot step. Figure 3 is just for illustration, Figure 4 is the actual constraint we use and the edges, vectors and distances are recorded in Table 2. One exception is for $(p_3^l)_{j+1}$ we would like to have

$$\begin{bmatrix}(p_3^l)_{j+1}^x & (p_3^l)_{j+1}^y\end{bmatrix}\begin{bmatrix}f_{j+1}^x - f_j^x \\ f_{j+1}^y - f_j^y\end{bmatrix} \geq (b_3^l)_{j+1} \text{ or } -\begin{bmatrix}(p_3^l)_{j+1}^x & (p_3^l)_{j+1}^y\end{bmatrix}\begin{bmatrix}f_{j+1}^x - f_j^x \\ f_{j+1}^y - f_j^y\end{bmatrix} \leq -(b_3^l)_{j+1},$$

where $(b_i^l)_{j+1} \geq 0$ is perpendicular distance between the support foot center and the i-th feasible region edge for the $(j+1)$-th foot step. Similarly we have

$$\begin{bmatrix}(p_i^r)_{j+1}^x & (p_i^r)_{j+1}^y\end{bmatrix}\begin{bmatrix}f_{j+1}^x - f_j^x \\ f_{j+1}^y - f_j^y\end{bmatrix} \leq (b_i^r)_{j+1}, \tag{27}$$

and

$$\begin{bmatrix}(p_3^r)_{j+1}^x & (p_3^r)_{j+1}^y\end{bmatrix}\begin{bmatrix}f_{j+1}^x - f_j^x \\ f_{j+1}^y - f_j^y\end{bmatrix} \geq (b_3^r)_{j+1} \text{ or } -\begin{bmatrix}(p_3^r)_{j+1}^x & (p_3^r)_{j+1}^y\end{bmatrix}\begin{bmatrix}f_{j+1}^x - f_j^x \\ f_{j+1}^y - f_j^y\end{bmatrix} \leq -(b_3^r)_{j+1}.$$

Note we have $(b_i^l)_{j+1} = (b_i^r)_{j+1}$. In matrix form we have

$$\begin{bmatrix}(p_1^l)_{j+1}^x & (p_1^l)_{j+1}^y \\ (p_2^l)_{j+1}^x & (p_2^l)_{j+1}^y \\ -(p_3^l)_{j+1}^x & -(p_3^l)_{j+1}^y \\ (p_4^l)_{j+1}^x & (p_4^l)_{j+1}^y \\ (p_5^l)_{j+1}^x & (p_5^l)_{j+1}^y\end{bmatrix}\begin{bmatrix}f_{j+1}^x - f_j^x \\ f_{j+1}^y - f_j^y\end{bmatrix} \leq \begin{bmatrix}(b_1^l)_{j+1} \\ (b_2^l)_{j+1} \\ -(b_3^l)_{j+1} \\ (b_4^l)_{j+1} \\ (b_5^l)_{j+1}\end{bmatrix},$$

and

$$\begin{bmatrix}(p_1^r)_{j+1}^x & (p_1^r)_{j+1}^y \\ (p_2^r)_{j+1}^x & (p_2^r)_{j+1}^y \\ -(p_3^r)_{j+1}^x & -(p_3^r)_{j+1}^y \\ (p_4^r)_{j+1}^x & (p_4^r)_{j+1}^y \\ (p_5^r)_{j+1}^x & (p_5^r)_{j+1}^y\end{bmatrix}\begin{bmatrix}f_{j+1}^x - f_j^x \\ f_{j+1}^y - f_j^y\end{bmatrix} \leq \begin{bmatrix}(b_1^r)_{j+1} \\ (b_2^r)_{j+1} \\ -(b_3^r)_{j+1} \\ (b_4^r)_{j+1} \\ (b_5^r)_{j+1}\end{bmatrix}$$

To write this constraint w.r.t. the decision variables $\begin{bmatrix}\dddot{X}_k & X_k^f & \dddot{Y}_k & Y_k^f\end{bmatrix}^T$ we have

$$\begin{bmatrix}(p_1^l)_{j+1}^x & (p_1^l)_{j+1}^y \\ (p_2^l)_{j+1}^x & (p_2^l)_{j+1}^y \\ -(p_3^l)_{j+1}^x & -(p_3^l)_{j+1}^y \\ (p_4^l)_{j+1}^x & (p_4^l)_{j+1}^y \\ (p_5^l)_{j+1}^x & (p_5^l)_{j+1}^y\end{bmatrix}\begin{bmatrix}f_{k+1}^x - f_k^x \\ f_{k+1}^y - f_k^y\end{bmatrix} \leq \begin{bmatrix}(b_1^l)_{j+1} \\ (b_2^l)_{j+1} \\ -(b_3^l)_{j+1} \\ (b_4^l)_{j+1} \\ (b_5^l)_{j+1}\end{bmatrix}$$

$$\begin{bmatrix}(p_1^l)_{j+1}^x & (p_1^l)_{j+1}^y \\ (p_2^l)_{j+1}^x & (p_2^l)_{j+1}^y \\ -(p_3^l)_{j+1}^x & -(p_3^l)_{j+1}^y \\ (p_4^l)_{j+1}^x & (p_4^l)_{j+1}^y \\ (p_5^l)_{j+1}^x & (p_5^l)_{j+1}^y\end{bmatrix}\begin{bmatrix}-1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1\end{bmatrix}\begin{bmatrix}f_k^x \\ f_{k+1}^x \\ f_k^y \\ f_{k+1}^y\end{bmatrix} \leq \begin{bmatrix}(b_1^l)_{j+1} \\ (b_2^l)_{j+1} \\ -(b_3^l)_{j+1} \\ (b_4^l)_{j+1} \\ (b_5^l)_{j+1}\end{bmatrix}.$$

We can simplify the inequality above by defining

$$\mathbf{G}^l_{j+1} = \begin{bmatrix} (p^l_1)^x_{j+1} & (p^l_1)^y_{j+1} \\ (p^l_2)^x_{j+1} & (p^l_2)^y_{j+1} \\ -(p^l_3)^x_{j+1} & -(p^l_3)^y_{j+1} \\ (p^l_4)^x_{j+1} & (p^l_4)^y_{j+1} \\ (p^l_5)^x_{j+1} & (p^l_5)^y_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{g}^{x,l}_{j+1} & \mathbf{g}^{y,l}_{j+1} \end{bmatrix} \text{ and } \mathbf{b}^l_{j+1} = \begin{bmatrix} (b^l_1)_{j+1} \\ (b^l_2)_{j+1} \\ -(b^l_3)_{j+1} \\ (b^l_4)_{j+1} \\ (b^l_5)_{j+1} \end{bmatrix}.$$

Therefore we have

$$\mathbf{G}^l_{j+1} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} f^x_j \\ f^x_{j+1} \\ f^y_j \\ f^y_{j+1} \end{bmatrix} \le \mathbf{b}^l_{j+1}.$$

Note that we have

$$(\mathbf{G}^l_{j+1})^y = -(\mathbf{G}^r_{j+1})^y \text{ and } \mathbf{b}^l_{j+1} = \mathbf{b}^r_{j+1},$$

therefore we use $\mathbf{b}_{j+1}$ instead of specifying whether it is w.r.t the left or right foot. If we consider the next two steps we have (assuming the next step is moving the left foot)

$$\begin{bmatrix} \mathbf{g}^{x,l}_{j+1} & 0 & \mathbf{g}^{y,l}_{j+1} & 0 \\ 0 & \mathbf{g}^{x,r}_{j+2} & 0 & \mathbf{g}^{y,r}_{j+2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} f^x_j \\ f^x_{j+1} \\ f^x_{j+2} \\ f^y_j \\ f^y_{j+1} \\ f^y_{j+2} \end{bmatrix} \le \begin{bmatrix} \mathbf{b}_{j+1} \\ \mathbf{b}_{j+2} \end{bmatrix},$$

if we define

$$M = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix},\ f^{x,y}_j = f^{x,y}_k,\ \{X,Y\}^f_k = \begin{bmatrix} f^{x,y}_{j+1} \\ f^{x,y}_{j+2} \end{bmatrix},$$

where $f^{x,y}_j$ is the current foot step position while also being the foot step position at the current time step thus we can change the subscript to represent the k-time step, $\{X,Y\}^f_k$ denotes the planned future foot steps at the k-th time step. Using this definition we have

$$\mathbf{G}_{j+1} M \begin{bmatrix} f^x_k \\ \tilde{F}^x_k \\ f^y_k \\ \tilde{F}^y_k \end{bmatrix} \le \mathbf{b}_{j+1}$$

$$\mathbf{G}_{j+1} M \left(\begin{bmatrix} 0 \\ X^f_k \\ 0 \\ Y^f_k \end{bmatrix} + \begin{bmatrix} f^x_k \\ \mathbf{0} \\ f^y_k \\ \mathbf{0} \end{bmatrix} \right) \le \mathbf{b}_{j+1}$$

$$\mathbf{G}_{j+1} M \begin{bmatrix} 0 \\ X^f_k \\ 0 \\ Y^f_k \end{bmatrix} \le \mathbf{b}_{j+1} - \mathbf{G}_{j+1} M \begin{bmatrix} f^x_k \\ \mathbf{0} \\ f^y_k \\ \mathbf{0} \end{bmatrix}$$

$$\mathbf{G}_{j+1} M \begin{bmatrix} 0 \\ X^f_k \\ 0 \\ Y^f_k \end{bmatrix} \le \mathbf{b}_{j+1} + \mathbf{G}_{j+1} \begin{bmatrix} f^x_k \\ 0 \\ f^y_k \\ 0 \end{bmatrix},$$

with $\mathbf{G}_{j+1} \in \mathbb{R}^{10\times 4}$ and $\mathbf{b}_{j+1} \in \mathbb{R}^{10\times 1}$ for a two step preview horizon. One observation to make is

$$
\begin{aligned}
M \begin{bmatrix} 0 \\ \tilde{F}_k^x \\ 0 \\ \tilde{F}_k^y \end{bmatrix} &= \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ X_k^f \\ 0 \\ Y_k^f \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{0}_{1\times N} & 1 & 0 & \mathbf{0}_{1\times N} & 0 & 0 \\ \mathbf{0}_{1\times N} & -1 & 1 & \mathbf{0}_{1\times N} & 0 & 0 \\ \mathbf{0}_{1\times N} & 0 & 0 & \mathbf{0}_{1\times N} & 1 & 0 \\ \mathbf{0}_{1\times N} & 0 & 0 & \mathbf{0}_{1\times N} & -1 & 1 \end{bmatrix} \begin{bmatrix} \dddot{X}_k \\ X_k^f \\ \dddot{Y}_k \\ Y_k^f \end{bmatrix} \\
&= \tilde{M} \begin{bmatrix} \dddot{X}_k \\ X_k^f \\ \dddot{Y}_k \\ Y_k^f \end{bmatrix},
\end{aligned}
$$

with $\tilde{M} \in \mathbb{R}^{4\times 2N+4}$ for a two step preview horizon. Using the associative property of matrix multiplication, $ABC = A(BC) = A(\tilde{B}\tilde{C})$ if $(BC) = (\tilde{B}\tilde{C})$, we can write the foot placement constraint w.r.t. support foot as

$$
\mathbf{G}_{j+1}\tilde{M} \begin{bmatrix} \dddot{X}_k \\ X_k^f \\ \dddot{Y}_k \\ Y_k^f \end{bmatrix} \leq \mathbf{b}_{j+1} + \mathbf{G}_{j+1} \begin{bmatrix} f_k^x \\ 0 \\ f_k^y \\ 0 \end{bmatrix} \tag{28}
$$

3.6.2 Foot Placement w.r.t. Support Foot for Initial Double Support Phase

Since the current and second foot step are already determined, all that needs to be constrained are the third and subsequent foot steps. Thus, by altering (28) we can have

$$
\mathbf{G}_{j+1}\tilde{M} \begin{bmatrix} \dddot{X}_k \\ X_1^f \\ \dddot{Y}_k \\ Y_1^f \end{bmatrix} \leq \mathbf{b}_{j+1} + \mathbf{G}_{j+1} \begin{bmatrix} f_1^x \\ 0 \\ f_1^y \\ 0 \end{bmatrix}. \tag{29}
$$

If we first lift the right foot we should have (29) as

$$
\begin{bmatrix} \mathbf{g}_{j+1}^{x,r} & 0 & \mathbf{g}_{j+1}^{y,r} & 0 \\ 0 & \mathbf{g}_{j+2}^{x,l} & 0 & \mathbf{g}_{j+2}^{y,l} \end{bmatrix} \begin{bmatrix} f_2^x - f_1^x \\ f_3^x - f_2^x \\ f_2^y - f_1^y \\ f_3^y - f_2^y \end{bmatrix} \leq \begin{bmatrix} \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}
$$

$$
\begin{bmatrix} \mathbf{g}_{j+1}^{x,r} & 0 & \mathbf{g}_{j+1}^{y,r} & 0 \\ 0 & \mathbf{g}_{j+2}^{x,l} & 0 & \mathbf{g}_{j+2}^{y,l} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} f_1^x \\ f_2^x \\ f_3^x \\ f_1^y \\ f_2^y \\ f_3^y \end{bmatrix} \leq \begin{bmatrix} \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}
$$

$$
\mathbf{G}_1 M \left(\begin{bmatrix} 0 \\ f_2^x \\ f_3^x \\ 0 \\ f_2^y \\ f_3^y \end{bmatrix} + \begin{bmatrix} f_1^x \\ 0 \\ 0 \\ f_1^y \\ 0 \\ 0 \end{bmatrix} \right) \leq \begin{bmatrix} \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}
$$

$$
\mathbf{G}_1 M \begin{bmatrix} 0 \\ f_2^x \\ f_3^x \\ 0 \\ f_2^y \\ f_3^y \end{bmatrix} \leq \begin{bmatrix} \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix} - \mathbf{G}_1 M \begin{bmatrix} f_1^x \\ 0 \\ 0 \\ f_1^y \\ 0 \\ 0 \end{bmatrix}.
$$

We have

$$\mathbf{G}_1 M \begin{bmatrix} f_1^x \\ 0 \\ 0 \\ f_1^y \\ 0 \\ 0 \end{bmatrix} = \mathbf{G}_1 \left(\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} f_1^x \\ 0 \\ 0 \\ f_1^y \\ 0 \\ 0 \end{bmatrix} \right) = -\mathbf{G}_1 \begin{bmatrix} f_1^x \\ 0 \\ f_1^y \\ 0 \end{bmatrix},$$

and

$$\begin{aligned} M \begin{bmatrix} 0 \\ f_2^x \\ f_3^x \\ 0 \\ f_2^y \\ f_3^y \end{bmatrix} &= \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ f_2^x \\ f_3^x \\ 0 \\ f_2^y \\ f_3^y \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0}_{1\times N} & 1 & 0 & \mathbf{0}_{1\times N} & 0 & 0 \\ \mathbf{0}_{1\times N} & -1 & 1 & \mathbf{0}_{1\times N} & 0 & 0 \\ \mathbf{0}_{1\times N} & 0 & 0 & \mathbf{0}_{1\times N} & 1 & 0 \\ \mathbf{0}_{1\times N} & 0 & 0 & \mathbf{0}_{1\times N} & -1 & 1 \end{bmatrix} \begin{bmatrix} \dddot{X}_1 \\ f_2^x \\ f_3^x \\ \dddot{Y}_1 \\ f_2^y \\ f_3^y \end{bmatrix} \\ &= \tilde{M} \begin{bmatrix} \dddot{X}_1 \\ f_2^x \\ f_3^x \\ \dddot{Y}_1 \\ f_2^y \\ f_3^y \end{bmatrix}. \end{aligned}$$

Therefore, the support feet constraint becomes

$$\mathbf{G}_1 \tilde{M} \begin{bmatrix} \dddot{X}_1 \\ f_2^x \\ f_3^x \\ \dddot{Y}_1 \\ f_2^y \\ f_3^y \end{bmatrix} \leq \begin{bmatrix} \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix} + \mathbf{G}_1 \begin{bmatrix} f_1^x \\ 0 \\ f_1^y \\ 0 \end{bmatrix}$$

3.6.3 Foot Placement w.r.t. Swing Foot

The constraint on the foot placement w.r.t. the swing foot simply ensures the next foot step is at a position that is feasible for the swing foot to reach given its current position and the maximum end-effector velocity moving forward

$$\begin{bmatrix} n_s^x & n_s^y \\ n_f^x & n_f^y \end{bmatrix} \begin{bmatrix} (X_k^f)_1 - x^f(t) \\ (Y_k^f)_1 - y^f(t) \end{bmatrix} \leq t_r v_{\max} = t_r \begin{bmatrix} \|v_s\|_{\max} \\ \|v_f\|_{\max} \end{bmatrix}, \tag{30}$$

where the next foot step position is $[(X_k^f)_1, (Y_k^f)_1]$, the current position of the swing foot is $[x^f(t), y^f(t)]$, the time remaining in this foot step cycle is t_r, the maximum end-effector forward velocity is $v_{\max}$, $n^{x,y}$ represent the vector of the velocity direction, where s represents the sagittal plane direction and f represents the frontal plane direction, $\|v_s\|_{\max}$ and $\|v_f\|_{\max}$ represents the maximum speed in the sagittal and frontal plane, respectively. Also writing in the form of decision variables $\begin{bmatrix} \dddot{X}_k & X_k^f & \dddot{Y}_k & Y_k^f \end{bmatrix}^T$ we have

$$\begin{aligned} \begin{bmatrix} n_s^x & n_s^y \\ n_f^x & n_f^y \end{bmatrix} \begin{bmatrix} (X_k^f)_1 - x^f(t) \\ (Y_k^f)_1 - y^f(t) \end{bmatrix} &\leq t_r \begin{bmatrix} \|v_s\|_{\max} \\ \|v_f\|_{\max} \end{bmatrix} \\ \begin{bmatrix} n_s^x & n_s^y \\ n_f^x & n_f^y \end{bmatrix} \begin{bmatrix} (X_k^f)_1 \\ (Y_k^f)_1 \end{bmatrix} &\leq t_r \begin{bmatrix} \|v_s\|_{\max} \\ \|v_f\|_{\max} \end{bmatrix} + \begin{bmatrix} n_s^x & n_s^y \\ n_f^x & n_f^y \end{bmatrix} \begin{bmatrix} x^f(t) \\ y^f(t) \end{bmatrix} \\ \begin{bmatrix} \mathbf{0}_{1\times N} & n_s^x & 0 & \mathbf{0}_{1\times N} & n_s^y & 0 \\ \mathbf{0}_{1\times N} & n_f^x & 0 & \mathbf{0}_{1\times N} & n_f^y & 0 \end{bmatrix} \begin{bmatrix} \dddot{X}_k \\ (X_k^f)_1 \\ (X_k^f)_2 \\ \dddot{Y}_k \\ (Y_k^f)_1 \\ (Y_k^f)_2 \end{bmatrix} &\leq t_r \begin{bmatrix} \|v_s\|_{\max} \\ \|v_f\|_{\max} \end{bmatrix} + \begin{bmatrix} n_s^x & n_s^y \\ n_f^x & n_f^y \end{bmatrix} \begin{bmatrix} x^f(t) \\ y^f(t) \end{bmatrix}. \end{aligned}$$

which gives us

$$
\begin{bmatrix} \mathbf{0}_{1\times N} & n_s^x & 0 & \mathbf{0}_{1\times N} & n_s^y & 0 \\ \mathbf{0}_{1\times N} & n_f^x & 0 & \mathbf{0}_{1\times N} & n_f^y & 0 \end{bmatrix} \begin{bmatrix} \dddot{X}_k \\ X_k^f \\ \dddot{Y}_k \\ Y_k^f \end{bmatrix} \leq t_r \begin{bmatrix} \|v_s\|_{\max} \\ \|v_f\|_{\max} \end{bmatrix} + \begin{bmatrix} n_s^x & n_s^y \\ n_f^x & n_f^y \end{bmatrix} \begin{bmatrix} x^f(t) \\ y^f(t) \end{bmatrix}, \tag{31}
$$

with

$$
\begin{bmatrix} \mathbf{0}_{1\times N} & n_s^x & 0 & \mathbf{0}_{1\times N} & n_s^y & 0 \\ \mathbf{0}_{1\times N} & n_f^x & 0 & \mathbf{0}_{1\times N} & n_f^y & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2N+4},
$$

for a two step preview horizon.

3.6.4 Foot Placement w.r.t. Swing Foot for Initial Double Support Phase

Since during the double support phase there is no swing foot, we would not require this constraint.

3.7 Cost Function

The cost function in use is from [2]

$$
\begin{aligned}
\min_{u_k} \frac{\alpha}{2}\|\dddot{X}_k\|^2 + \frac{\beta}{2}\|\dot{X}_{k+1} - \dot{X}_{k+1}^{\text{ref}}\|^2 + \frac{\gamma}{2}\|Z_{k+1}^x - Z_{k+1}^{x_{\text{ref}}}\|^2 \\
+ \frac{\alpha}{2}\|\dddot{Y}_k\|^2 + \frac{\beta}{2}\|\dot{Y}_{k+1} - \dot{Y}_{k+1}^{\text{ref}}\|^2 + \frac{\gamma}{2}\|Z_{k+1}^y - Z_{k+1}^{y_{\text{ref}}}\|^2.
\end{aligned} \tag{32}
$$

To understand the meaning of (32) we can separate it into three parts. The first part

$$
\min_{u_k} \frac{\beta}{2}\|\dot{X}_{k+1} - \dot{X}_{k+1}^{\text{ref}}\|^2 + \frac{\beta}{2}\|\dot{Y}_{k+1} - \dot{Y}_{k+1}^{\text{ref}}\|^2
$$

is aimed to regulate the CoM speed to a desired mean value $(\dot{x}^{\text{ref}}, \dot{y}^{\text{ref}})$. In [2], it is stated that by simply optimizing over the regulated velocity cost stable walking can be achieved. The second part

$$
\min_{u_k} \frac{\alpha}{2}\|\dddot{X}_k\|^2 + \frac{\alpha}{2}\|\dddot{Y}_k\|^2,
$$

minimizes the jerks, which helps, when weakly weighted (small α), smooth the contact forces and therefore the resulting motion. The third part

$$
\min_{u_k} \frac{\gamma}{2}\|Z_{k+1}^x - Z_{k+1}^{x_{\text{ref}}}\|^2 + \frac{\gamma}{2}\|Z_{k+1}^y - Z_{k+1}^{y_{\text{ref}}}\|^2,
$$

makes the CoP track a reference CoP, which is calculated using (15). When weakly weighted, centering the CoP in the foot allows for faster and more robust reactions to changes in the state of the system or in the desired speed of CoM. Therefore, the CoP references $Z_{k+1}^{x_{\text{ref}}}$ and $Z_{k+1}^{y_{\text{ref}}}$ are simply the foot step location, which is what (15) calculates.

3.8 Quadratic Program

To write the optimization in the canonical form

$$
\min_{u_k} \quad \frac{1}{2}u_k^T Q_k u_k + p_k^T u_k \tag{33a}
$$

$$
G_k u_k \leq h_k \tag{33b}
$$

where the constraints are given in (19), (28) and (31). Let's first write the x component of (32) in a matrix form. The x component itself can be separated into three parts: jerk, velocity and CoP. We will write each of the three parts in matrix form

$$
\begin{aligned}
\frac{\alpha}{2}\|\dddot{X}_k\|^2 &= \frac{1}{2}\alpha \dddot{X}_k^T \dddot{X}_k \\
\frac{\beta}{2}\|\dot{X}_{k+1} - \dot{X}_{k+1}^{\text{ref}}\|^2 &= \frac{1}{2}\beta(\dot{X}_{k+1} - \dot{X}_{k+1}^{\text{ref}})^T(\dot{X}_{k+1} - \dot{X}_{k+1}^{\text{ref}}) \\
&= \frac{1}{2}\beta(S_v\hat{x}_k + U_v\dddot{X}_k - \dot{X}_{k+1}^{\text{ref}})^T(S_v\hat{x}_k + U_v\dddot{X}_k - \dot{X}_{k+1}^{\text{ref}}) \\
&= \frac{1}{2}\beta(\hat{x}_k^T S_v^T + \dddot{X}_k^T U_v^T - (\dot{X}_{k+1}^{\text{ref}})^T)(S_v\hat{x}_k + U_v\dddot{X}_k - \dot{X}_{k+1}^{\text{ref}}),
\end{aligned}
$$

note that we are minimizing over the decision variables $\dddot{X}_k$ and X_k^f, therefore any term that does not contain a decision variables can be seen equivalent to zero to the optimization algorithm. With this in mind, we can write the above polynomial in an optimization equivalent form

$$\begin{aligned}
&\frac{1}{2}\beta(\hat{x}_k^T S_v^T + \dddot{X}_k^T U_v^T - (\dot{X}_{k+1}^{\text{ref}})^T)(S_v\hat{x}_k + U_v\dddot{X}_k - \dot{X}_{k+1}^{\text{ref}}) \\
\text{equivalent to}\quad &\frac{1}{2}\beta\Big(\hat{x}_k^T S_v^T U_v \dddot{X}_k + \dddot{X}_k^T U_v^T S_v \hat{x}_k + \dddot{X}_k^T U_v^T U_v \dddot{X}_k - \dddot{X}_k^T U_v^T \dot{X}_{k+1}^{\text{ref}} - (\dot{X}_{k+1}^{\text{ref}})^T U_v \dddot{X}_k\Big) \\
&= \frac{1}{2}\beta\Big(\dddot{X}_k^T U_v^T U_v \dddot{X}_k + \dddot{X}_k^T (U_v^T S_v \hat{x}_k - U_v^T \dot{X}_{k+1}^{\text{ref}}) + (\hat{x}_k^T S_v^T U_v - (\dot{X}_{k+1}^{\text{ref}})^T U_v)\dddot{X}_k\Big),
\end{aligned}$$

since both $(\hat{x}_k^T S_v^T U_v - (\dot{X}_{k+1}^{\text{ref}})^T U_v)\dddot{X}_k$ and $\dddot{X}_k^T (U_v^T S_v \hat{x}_k - U_v^T \dot{X}_{k+1}^{\text{ref}})$ are scalars and we can see that they are the transpose of each other we can say that

$$\frac{\beta}{2}\|\dot{X}_{k+1} - \dot{X}_{k+1}^{\text{ref}}\|^2 \text{ is equivalent to } \frac{1}{2}\beta \dddot{X}_k^T U_v^T U_v \dddot{X}_k + \beta(\hat{x}_k^T S_v^T - (\dot{X}_{k+1}^{\text{ref}})^T) U_v \dddot{X}_k.$$

For the CoP term we have

$$\begin{aligned}
&\frac{\gamma}{2}\|Z_{k+1}^x - Z_{k+1}^{x_{\text{ref}}}\|^2 \\
&= \frac{\gamma}{2}\|Z_{k+1}^x - U_{k+1}^c X_k^{fc} - U_{k+1} X_k^f\|^2 \\
&= \frac{\gamma}{2}\|S_z\hat{x}_k + U_z\dddot{X}_k - U_{k+1}^c X_k^{fc} - U_{k+1} X_k^f\|^2 \\
&= \frac{\gamma}{2}(S_z\hat{x}_k + U_z\dddot{X}_k - U_{k+1}^c X_k^{fc} - U_{k+1} X_k^f)^T (S_z\hat{x}_k + U_z\dddot{X}_k - U_{k+1}^c X_k^{fc} - U_{k+1} X_k^f) \\
&= \frac{\gamma}{2}(\hat{x}_k^T S_z^T + \dddot{X}_k^T U_z^T - (X_k^{fc})^T (U_{k+1}^c)^T - (X_k^f)^T U_{k+1}^T)(S_z\hat{x}_k + U_z\dddot{X}_k - U_{k+1}^c X_k^{fc} - U_{k+1} X_k^f).
\end{aligned}$$

As before we need to get rid of the term that does not contain any decision variables

$$\begin{aligned}
&= \frac{\gamma}{2}\Big[\hat{x}_k^T S_z^T (U_z\dddot{X}_k - U_{k+1}X_k^f) + \dddot{X}_k^T U_z^T (S_z\hat{x}_k + U_z\dddot{X}_k - U_{k+1}^c X_k^{fc} - U_{k+1}X_k^f) \\
&\quad - (X_k^{fc})^T (U_{k+1}^c)^T (U_z\dddot{X}_k - U_{k+1}X_k^f) - (X_k^f)^T U_{k+1}^T (S_z\hat{x}_k + U_z\dddot{X}_k - U_{k+1}^c X_k^{fc} - U_{k+1}X_k^f)\Big] \\
&= \frac{\gamma}{2}\Big[\dddot{X}_k^T U_z^T U_z \dddot{X}_k + (X_k^f)^T U_{k+1}^T U_{k+1} X_k^f + \dddot{X}_k^T U_z^T (S_z\hat{x} - U_{k+1}^c X_k^{fc}) \\
&\quad + (\hat{x}_k^T S_z^T - (X_k^{fc})^T (U_{k+1}^c)^T) U_z \dddot{X}_k - (X_k^f)^T U_{k+1}^T (S_z\hat{x}_k - U_{k+1}^c X_k^{fc}) \\
&\quad + ((X_k^{fc})^T (U_{k+1}^c)^T - \hat{x}_k^T S_z^T) U_{k+1} X_k^f - \dddot{X}_k^T U_z^T U_{k+1} X_k^f - (X_k^f)^T U_{k+1}^T U_z \dddot{X}_k\Big].
\end{aligned}$$

Also we can group the scalar terms which gives us

$$\begin{aligned}
&\frac{\gamma}{2}\Big(\dddot{X}_k^T U_z^T U_z \dddot{X}_k + (X_k^f)^T U_{k+1}^T U_{k+1} X_k^f - \dddot{X}_k^T U_z^T U_{k+1} X_k^f - (X_k^f)^T U_{k+1}^T U_z \dddot{X}_k\Big) \\
&+ \gamma(\hat{x}_k^T S_z^T - (X_k^{fc})^T (U_{k+1}^c)^T) U_z \dddot{X}_k - (\hat{x}_k^T S_z^T - (X_k^{fc})^T (U_{k+1}^c)^T) U_{k+1} X_k^f
\end{aligned}$$

Thus, in QP form we can write the components explicitly as

$$Q_k = \begin{bmatrix} Q_k' & 0 \\ 0 & Q_k' \end{bmatrix} \tag{34}$$

$$Q_k' = \begin{bmatrix} \alpha I + \beta U_v^T U_v + \gamma U_z^T U_z & -\gamma U_z^T U_{k+1} \\ -\gamma U_{k+1}^T U_z & \gamma U_{k+1}^T U_{k+1} \end{bmatrix} \tag{35}$$

$$p_k = \begin{bmatrix} \beta U_v^T (S_v\hat{x}_k - \dot{X}_{k+1}^{\text{ref}}) + \gamma U_z^T (S_z\hat{x}_k - U_{k+1}^c X_k^{fc}) \\ -\gamma U_{k+1}^T (S_z\hat{x}_k - U_{k+1}^c X_k^{fc}) \\ \beta U_v^T (S_v\hat{y}_k - \dot{Y}_{k+1}^{\text{ref}}) + \gamma U_z^T (S_z\hat{y}_k - U_{k+1}^c Y_k^{fc}) \\ -\gamma U_{k+1}^T (S_z\hat{y}_k - U_{k+1}^c Y_k^{fc}) \end{bmatrix} \tag{36}$$

$$G_k = \begin{bmatrix} D_{k+1}\begin{bmatrix} U_z & -U_{k+1} & 0 & 0 \\ 0 & 0 & U_z & -U_{k+1} \end{bmatrix} \\ \mathbf{G}_{j+1}\tilde{M} \\ \begin{bmatrix} \mathbf{0}_{1\times N} & n_s^x & 0 & \mathbf{0}_{1\times N} & n_s^y & 0 \\ \mathbf{0}_{1\times N} & n_f^x & 0 & \mathbf{0}_{1\times N} & n_f^y & 0 \end{bmatrix} \end{bmatrix} \tag{37}$$

$$h_k = \begin{bmatrix} b_{k+1} + D_{k+1} \begin{bmatrix} U^c_{k+1} X^{fc}_k - S_z \hat{x}_k \\ U^c_{k+1} Y^{fc}_k - S_z \hat{y}_k \end{bmatrix} \\ \mathbf{b}_{j+1} + \mathbf{G}_{j+1} \begin{bmatrix} f^x_k \\ 0 \\ f^y_k \\ 0 \end{bmatrix} \\ t_r \begin{bmatrix} \|v_s\|_{\max} \\ \|v_f\|_{\max} \end{bmatrix} + \begin{bmatrix} n^x_s & n^y_s \\ n^x_f & n^y_f \end{bmatrix} \begin{bmatrix} x^f(t) \\ y^f(t) \end{bmatrix} \end{bmatrix}. \tag{38}$$

3.8.1 Cost function for Initial Double Support Phase

The only difference here is for the CoP part, which changes to

$$\begin{aligned} &\frac{\gamma}{2}\|Z^x_{k+1} - Z^{x_{\text{ref}}}_{k+1}\|^2 \\ =& \frac{\gamma}{2}\|Z^x_{k+1} - U^c_{k+1} X^{fc}_k - U_{k+1} X^f_k\|^2 \\ =& \frac{\gamma}{2}\|Z^x_{k+1} - U^c_{k+1} X^{fc}_0 - U_{k+1}\texttt{[:, :1]} X^{fc}_1 - U_{k+1}\texttt{[:, 1:]} X^f_1\|^2 \\ =& \frac{\gamma}{2}\|S_z \hat{x}_k + U_z \dddot{X}_k - U^c_{k+1} X^{fc}_0 - U_{k+1}\texttt{[:, :1]} X^{fc}_1 - U_{k+1}\texttt{[:, 1:]} X^f_1\|^2 \end{aligned}$$

let $a = S_z \hat{x}_k - U^c_{k+1} X^{fc}_0 - U_{k+1}\texttt{[:, :1]} X^{fc}_1$ and we have the above equation as

$$\begin{aligned} &\frac{\gamma}{2}\|S_z \hat{x}_k + U_z \dddot{X}_k - U^c_{k+1} X^{fc}_0 - U_{k+1}\texttt{[:, :1]} X^{fc}_1 - U_{k+1}\texttt{[:, 1:]} X^f_1\|^2 \\ =& \frac{\gamma}{2}\|a + U_z \dddot{X}_k - U_{k+1}\texttt{[:, 1:]} X^f_1\|^2 \\ =& \frac{\gamma}{2}(a + U_z \dddot{X}_k - U_{k+1}\texttt{[:, 1:]} X^f_1)^T (a + U_z \dddot{X}_k - U_{k+1}\texttt{[:, 1:]} X^f_1) \\ =& \frac{\gamma}{2}(a^T + \dddot{X}^T_k U^T_z - (X^f_1)^T U_{k+1}\texttt{[:, 1:]}^T)(a + U_z \dddot{X}_k - U_{k+1}\texttt{[:, 1:]} X^f_1) \end{aligned}$$

similar as before we can get rid of the scalar term that has nothing to do with the decision variables $\dddot{X}_k$ and X^f_1

$$\begin{aligned} &\frac{\gamma}{2}(a^T + \dddot{X}^T_k U^T_z - (X^f_1)^T U_{k+1}\texttt{[:, 1:]}^T)(a + U_z \dddot{X}_k - U_{k+1}\texttt{[:, 1:]} X^f_1) \\ =& \frac{\gamma}{2}\Big[a^T (U_z \dddot{X}_k - U_{k+1}\texttt{[:, 1:]} X^f_1) + \dddot{X}^T_k U^T_z (a + U_z \dddot{X}_k - U_{k+1}\texttt{[:, 1:]} X^f_1) \\ &- (X^f_1)^T U_{k+1}\texttt{[:, 1:]}^T (a + U_z \dddot{X}_k - U_{k+1}\texttt{[:, 1:]} X^f_1)\Big] \\ =& \frac{\gamma}{2}\Big[\dddot{X}^T_k U^T_z U_z \dddot{X}_k + (X^f_1)^T U_{k+1}\texttt{[:, 1:]}^T U_{k+1}\texttt{[:, 1:]} X^f_1 + 2a^T U_z \dddot{X}_k - 2a^T U_{k+1}\texttt{[:, 1:]} X^f_1 \\ &- \dddot{X}^T_k U^T_z U_{k+1}\texttt{[:, 1:]} X^f_1 - (X^f_1)^T U_{k+1}\texttt{[:, 1:]}^T U_z \dddot{X}_k\Big]. \end{aligned}$$

Along with

$$\frac{1}{2}\beta \dddot{X}^T_k U^T_v U_v \dddot{X}_k + \beta(\hat{x}^T_k S^T_v - (\dot{X}^{\text{ref}}_{k+1})^T) U_v \dddot{X}_k \text{ and } \frac{1}{2}\alpha \dddot{X}^T_k \dddot{X}_k,$$

we can have

$$Q'_k = \begin{bmatrix} \alpha I + \beta U^T_v U_v + \gamma U^T_z U_z & -\gamma U^T_z U_{k+1}\texttt{[:, 1:]} \\ -\gamma U_{k+1}\texttt{[:, 1:]}^T U_z & \gamma U_{k+1}\texttt{[:, 1:]}^T U_{k+1}\texttt{[:, 1:]} \end{bmatrix}$$

$$p_k = \begin{bmatrix} \beta U^T_v (S_v \hat{x}_k - \dot{X}^{\text{ref}}_{k+1}) + \gamma U^T_z (S_z \hat{x}_k - U_{k+1}\texttt{[:, :1]} X^{fc}_1) \\ -\gamma U_{k+1}\texttt{[:, 1:]}^T (S_z \hat{x}_k - U_{k+1}\texttt{[:, :1]} X^{fc}_1) \\ \beta U^T_v (S_v \hat{y}_k - \dot{Y}^{\text{ref}}_{k+1}) + \gamma U^T_z (S_z \hat{y}_k - U_{k+1}\texttt{[:, :1]} Y^{fc}_1) \\ -\gamma U_{k+1}\texttt{[:, 1:]}^T (S_z \hat{y}_k - U_{k+1}\texttt{[:, :1]} Y^{fc}_1) \end{bmatrix}$$

$$G = \begin{bmatrix} D_{k+1} \begin{bmatrix} U_z & -U_{k+1}\texttt{[:, 1:]} & 0 & 0 \\ 0 & 0 & U_z & -U_{k+1}\texttt{[:, 1:]} \end{bmatrix} \\ \mathbf{G}_{j+1} \tilde{M} \end{bmatrix}$$

Table 2: **Function Inputs**

	N	dt	h	g	tPf	m
U_p	✓	✓				
U_v	✓	✓				
U_a	✓	✓				
U_z	✓	✓	✓	✓		
S_p	✓	✓				
S_v	✓	✓				
S_a	✓	✓				
S_z	✓	✓	✓	✓		
U^c_{k+1}	✓				✓	✓
U_{k+1}	✓				✓	✓

$$h = \begin{bmatrix} b_{k+1} + D_{k+1} \begin{bmatrix} U^c_{k+1} X^{fc}_0 + U_{k+1}\texttt{[:, :1]} X^{fc}_1 - S_z \hat{x}_k \\ U^c_{k+1} Y^{fc}_0 + U_{k+1}\texttt{[:, :1]} Y^{fc}_1 - S_z \hat{y}_k \end{bmatrix} \\ \mathbf{b}_{j+1} + \mathbf{G}_{j+1} \begin{bmatrix} f^x_1 \\ 0 \\ f^y_1 \\ 0 \end{bmatrix} \end{bmatrix}$$

3.9 Forward Walking Experiment Details

At the beginning the left and right foot are placed $0.3m$ apart, while the CoM is at positioned at $(0, 0)$. Therefore, the left foot is positioned at $(0, 0.15)$ and the right foot is positioned at $(0, -0.15)$. We use a reference speed of $v_x = 0.3m/s$ and $v_y = 0.0m/s$.

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